Quotient Hopf-Galois structures and associated orders in Hopf-Galois extensions

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Overview

It is sometimes possible to relate questions about integral module structure in a Galois extension of local or global fields to analogous questions about subextensions.

In this talk we generalize these ideas to Hopf-Galois extensions.

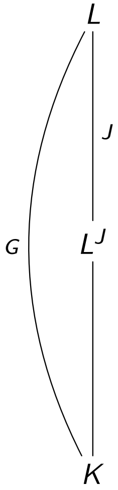
- Normality in Galois extensions via group algebras.
- Normality in separable Hopf-Galois extensions of fields.
- Two useful lemmas in Galois module theory.
- Hopf-Galois generalizations of these, and applications.

Normality in Galois extensions via group algebras

Let L/K be a Galois extension of fields with group G.
L/K is Hopf-Galois for $K[G]$.
We can characterize fixed fields via Hopf subalgebras:
The Hopf subalgebras of $K[G]$ are $K[J]$, with J a subgroup of G, and

$$\begin{aligned} L^J &= \{x \in L \mid \gamma(x) = x \text{ for all } \gamma \in J\} \\ &= \{x \in L \mid z \cdot x = \varepsilon(z)x \text{ for all } z \in K[J]\} \\ &= L^{K[J]}, \text{say.} \end{aligned}$$

L/L^J is Hopf-Galois for $L^J \otimes_K K[J]$.

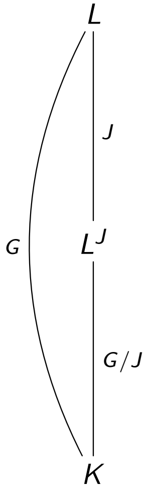


Normality in Galois extensions via group algebras

If J is a normal subgroup of G then L^J/K is a Galois extension with Galois group G/J.
In this case L^J/K is Hopf-Galois for $K[G/J]$.

Idea

Investigate analogous questions for Hopf-Galois structures on separable extensions of fields.



The Greither-Pareigis classification

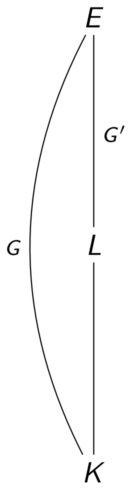
Theorem (Greither and Pareigis, 1987)

Let L/K be a separable extension of fields with Galois closure E.

- *Let $G = Gal(E/K)$, $G' = Gal(E/L)$, $X = G/G'$.*
- *Define $\lambda : G \to Perm(X)$ by $\lambda(\sigma)[\tau G'] = \sigma\tau G'$.*
- *Let G act on $Perm(X)$ by ${}^{\sigma}\eta = \lambda(\sigma)\eta\lambda(\sigma)^{-1}$.*

Then

- *There is a bijection between G-stable regular subgroups of $Perm(X)$ and Hopf-Galois structures on L/K;*
- *the Hopf algebra giving the Hopf-Galois structure corresponding to N is $E[N]^G$.*



Hopf subalgebras and fixed fields

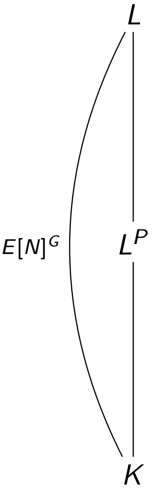
Let L/K be separable and Hopf-Galois for $E[N]^G$.

The Hopf subalgebras of $E[N]^G$ are $E[P]^G$ with P a G-stable subgroup of N.

Each Hopf subalgebra has a corresponding fixed field:

$$L^P = \{x \in L \mid z \cdot x = \varepsilon(z)x \text{ for all } z \in E[P]^G\}.$$

L/L^P is Hopf-Galois for $L^P \otimes_K E[P]^G$.



Example

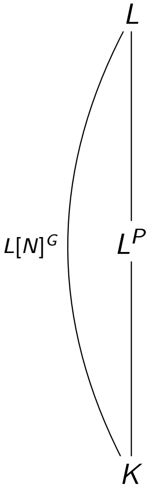
If L/K is Galois with group G then $K[G]$ corresponds to $\rho(G) \subset \mathrm{Perm}(G)$. The action of G on $\rho(G)$ is trivial, so every subgroup of $\rho(G)$ is G-stable. We recover the situation considered earlier.

Normality and quotient Hopf-Galois structures

Theorem (Koch, Kohl, T, Underwood, 2019)

Suppose that L/K is a Galois extension of fields that is Hopf-Galois for $L[N]^G$, and that P is a normal G-stable subgroup of N.
Then L^P/K is Hopf-Galois for $L[N/P]^G$.

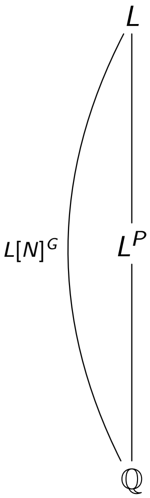
Important to note that L^P/K might not be Galois.



Normality and quotient Hopf-Galois structures

Example

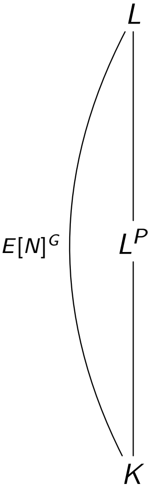
- Let L be the splitting field of $x^3 - 2$ over $\mathbb{Q}$.
- $L/\mathbb{Q}$ is Galois with Galois group $G \cong D_3$.
- $\mathrm{Perm}(G)$ contains G-stable regular subgroups that are isomorphic to C_6. Let N be one.
- $L/\mathbb{Q}$ is Hopf-Galois for $L[N]^G$.
- N has a unique subgroup P of order 2.
- P is normal and G-stable.
- By the theorem, $L^P/\mathbb{Q}$ is Hopf-Galois for $L[N/P]^G$.
- But $L^P/\mathbb{Q}$ is not Galois.



A slight generalization

Theorem

Suppose that L/K is a separable extension of fields that is Hopf-Galois for $E[N]^G$, and that P is a normal G-stable subgroup of N.
Then L^P/K is Hopf-Galois for $E[N/P]^G$.



Remainder of the talk is about the application of these ideas to questions of integral module structure.

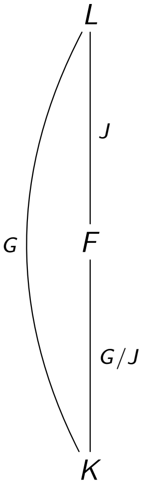
Henceforth, suppose that L/K is an extension of number fields or p-adic fields.

A useful lemma in Galois module theory

Suppose that L/K is Galois with group G, and $J \triangleleft G$.

Write $F = L^J$, and let

- $\pi : K[G] \twoheadrightarrow K[G/J]$ be the algebra homomorphism induced by the natural map $G \twoheadrightarrow G/J$;
- $\mathfrak{A}_{L/K}$ be the associated order of $\mathfrak{O}_L$ in $K[G]$;
- $\mathfrak{A}_{F/K}$ be the associated order of $\mathfrak{O}_F$ in $K[G/J]$.



Lemma (Byott and Lettl, 1996)

Suppose that $\mathfrak{O}_L = \mathfrak{A}_{L/K} \cdot \alpha$ and that L/F is (at most) tamely ramified. Then

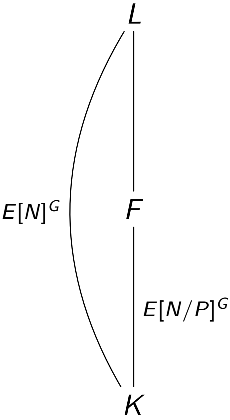
- $\mathfrak{A}_{F/K} = \pi(\mathfrak{A}_{L/K})$;
- $\mathfrak{O}_F = \mathfrak{A}_{F/K} \cdot \mathrm{Tr}_{L/F}(\alpha)$.

A Hopf-Galois version

Suppose that L/K is separable and Hopf-Galois for $E[N]^G$, and that $P \triangleleft N$ is G-stable.

Write $F = L^P$, and let

- $\mathfrak{A}_{L/K}$ be the associated order of $\mathfrak{O}_L$ in $E[N]^G$;
- $\mathfrak{A}_{F/K}$ be the associated order of $\mathfrak{O}_F$ in $E[N/P]^G$.



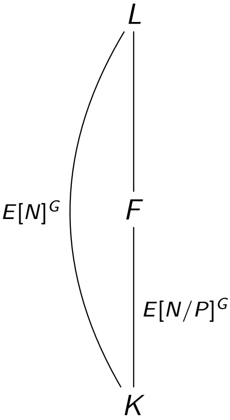
Lemma

The E-algebra homomorphism $\pi : E[N] \twoheadrightarrow E[N/P]$ induced by the natural map $N \twoheadrightarrow N/P$ descends to a K-algebra homomorphism $\pi : E[N]^G \twoheadrightarrow E[N/P]^G$

A Hopf-Galois version

Recall

- $F = L^P$;
- $\mathfrak{A}_{L/K}$ is the associated order of $\mathfrak{O}_L$ in $E[N]^G$;
- $\mathfrak{A}_{F/K}$ is the associated order of $\mathfrak{O}_F$ in $E[N/P]^G$.



Lemma

Suppose that $\mathfrak{O}_L = \mathfrak{A}_{L/K} \cdot \alpha$ and that L/F is tamely ramified. Then

- $\mathfrak{A}_{F/K} = \pi(\mathfrak{A}_{L/K})$;
- $\mathfrak{O}_F = \mathfrak{A}_{F/K} \cdot \mathrm{Tr}_{L/F}(\alpha)$.

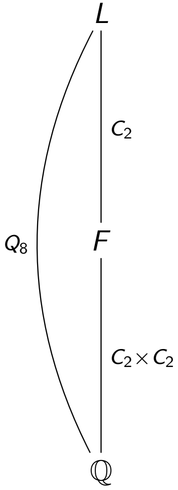
An application

Theorem (Taylor)

Let $L/\mathbb{Q}$ be a tamely ramified Galois extension with group $G \cong Q_8$, and suppose that $L/\mathbb{Q}$ is Hopf-Galois for $L[N]^G$ with N cyclic.
Then $\mathfrak{O}_L$ is locally free, but not free, over $\mathfrak{A}_{L/\mathbb{Q}}$.

Proof.

- Local freeness is already known.
- N has a unique subgroup P of order 2.
- P is normal and G-stable.
- $F = L^P$ is a real biquadratic extension of $\mathbb{Q}$.
- N/P is cyclic, and $K/\mathbb{Q}$ is Hopf-Galois for $L[N/P]^G$.

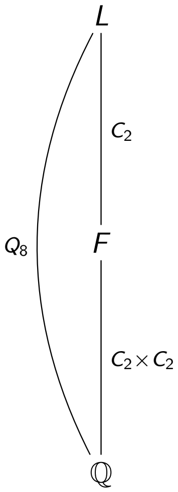


An application

Proof Continued...

- There are three HGS on $F/\mathbb{Q}$ for which the underlying N is cyclic.
- They correspond to the three quadratic subfields.
- $\mathfrak{O}_F$ is free over its associated order in a HGS only if the corresponding quadratic subfield is imaginary.
- Therefore $\mathfrak{O}_F$ is not free over its associated order in $L[N/P]^G$.
- By the lemma, $\mathfrak{O}_L$ is not free over $\mathfrak{A}_{L/\mathbb{Q}}$.

□



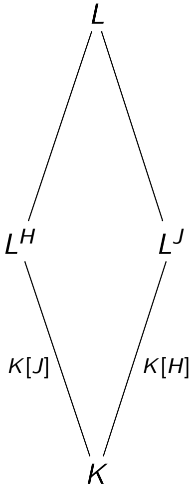
Another useful lemma in Galois module theory

Lemma (Byott and Lettl, 1996)

Suppose that L/K is Galois with group $G = H \times J$. Then L^H/K and L^J/K are linearly disjoint, and $L = L^H L^J$. Suppose in addition that

- $\mathfrak{d}(L^H/K)$ *and* $\mathfrak{d}(L^J/K)$ *are coprime;*
- $\mathfrak{O}_{L^H} = \mathfrak{A}_{L^H/K} \cdot \alpha$;
- $\mathfrak{O}_{L^J} = \mathfrak{A}_{L^J/K} \cdot \beta$.

Then $\mathfrak{O}_L = \mathfrak{A}_{L/K} \cdot \alpha\beta$.



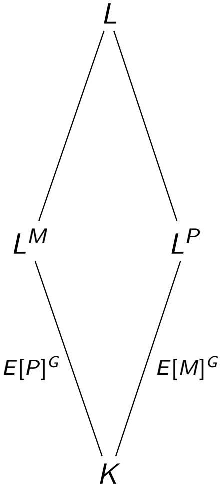
A Hopf-Galois version

Lemma

Suppose that L/K is separable and Hopf-Galois for $E[N]^G$, and that $N = M \times P$ for G-stable subgroups M, P of N. Then

- *L^M/K and L^P/K are linearly disjoint;*
- *L^P/K is Hopf-Galois for $E[M]^G$;*
- *L^M/K is Hopf-Galois for $E[P]^G$.*

Continued...



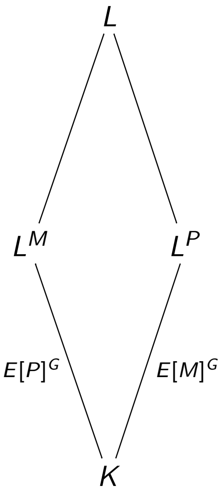
A Hopf-Galois version

Lemma (Continued)

Suppose in addition that

- $\mathfrak{d}(L^M/K)$ *and* $\mathfrak{d}(L^P/K)$ *are coprime;*
- $\mathfrak{O}_{L^M} = \mathfrak{A}_{L^M/K} \cdot \alpha$;
- $\mathfrak{O}_{L^P} = \mathfrak{A}_{L^P/K} \cdot \beta$.

Then $\mathfrak{O}_L = \mathfrak{A}_{L/K} \cdot \alpha\beta$.



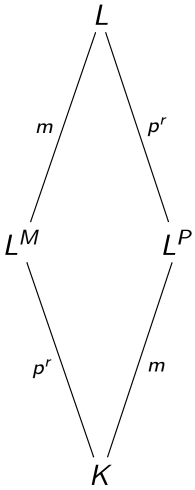
An application

Theorem

Let L/K be a tame separable extension of p-adic fields which is Hopf-Galois for $E[N]^G$, with N abelian. Then $\mathfrak{O}_L$ is a free $\mathfrak{A}_{L/K}$-module.

Proof.

- Write $N = M \times P$ with $|M| = m$, $|P| = p^r$, and $p \nmid m$.
- M, P are normal and G-stable.
- L^P/K is Hopf-Galois for $E[M]^G$.
- L^M/K is Hopf-Galois for $E[P]^G$.

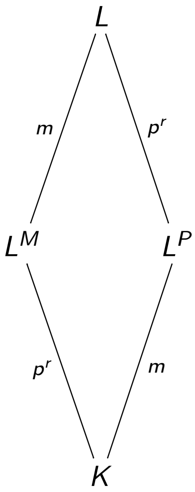


An application

Proof Continued..

- L^M/K is unramified, so $\mathfrak{O}_{L^M}$ is free over $\mathfrak{A}_{L^M/K}$.
- The degree of L^P/K is prime to p, so $\mathfrak{O}_{L^P}$ is free over $\mathfrak{A}_{L^P/K}$.
- $\mathfrak{d}(L^M/K)$ and $\mathfrak{d}(L^P/K)$ are coprime.
- By the lemma, $\mathfrak{O}_L$ is free over $\mathfrak{A}_{L/K}$.

□



Thank you for your attention.